Modal Logic

◇A	=	It's possible that A.
	=	A is true in some possible world.
A	=	It's true that A.
	=	A is true in the actual world.
☐A	=	It's necessary that A.
	=	A is true in all possible worlds.

The result of writing "◇" or "☐," and then a wff, is a wff.

A is possible (consistent, could be true) = $\Diamond A$

A is necessary (must be true, has to be true) = $\Box A$

A is impossible (self-contradictory)	=	$\sim\Diamond A$	=	A couldn't be true.
		$\Box\sim A$	=	A has to be false.

A is consistent (compatible) with B	=	It's possible that A and B are both true.
	=	$\Diamond(A \cdot B)$
A entails B	=	It's necessary that if A then B.
	=	$\Box(A \supset B)$
A is a contingent statement	=	A is possible and not-A is possible.
	=	$(\Diamond A \cdot \Diamond{\sim}A)$
A is a contingent truth	=	A is true but could have been false.
	=	$(A \cdot \Diamond{\sim}A)$

It's usually good to mimic the English word order:

necessary not	=	$\Box\sim$	necessary if	=	$\Box($
not necessary	=	$\sim\Box$	if necessary	=	$(\Box$

Each "necessary" or "possible" uses a separate box or diamond:

If A is necessary and B is possible, then C is possible = $((\Box A \cdot \Diamond B) \supset \Diamond C)$

This ambiguous sentence could have either of two meanings:

"If you're a bachelor, then you must be unmarried."

Simple Necessity	**Conditional Necessity**
$(B \supset \Box U)$	$\Box(B \supset U)$
If you're a bachelor, then you're *inherently unmarriable* (in no possible world would anyone ever marry you).	It's necessary that *if* you're a bachelor *then* you're unmarried.
If B, then U (by itself) is necessary.	It's necessary that if-B-then-U.

These forms aren't ambiguous:

If A, then B (by itself) is necessary	=	$(A \supset \Box B)$
If A, then B is inherently necessary	=	$(A \supset \Box B)$
A entails B	=	$\Box(A \supset B)$
Necessarily, if A then B	=	$\Box(A \supset B)$
It's necessary that if A then B	=	$\Box(A \supset B)$
"If A then B" is a necessary truth	=	$\Box(A \supset B)$

These forms are ambiguous:

"*If A, then it's necessary (must be) that B*" could mean "$(A \supset \Box B)$" or "$\Box(A \supset B)$."

"*If A, then it's impossible (couldn't be) that B*" could mean "$(A \supset \Box{\sim}B)$" or "$\Box(A \supset {\sim}B)$."

A is possible (could be true) = $\Diamond A$

A is necessary (must be true) = $\Box A$

A is impossible (self-contradictory) = $\sim\Diamond A$ = $\Box\sim A$

A is consistent with B = $\Diamond(A \cdot B)$

A entails B = $\Box(A \supset B)$

A is a contingent statement = $(\Diamond A \cdot \Diamond\sim A)$

A is a contingent truth = $(A \cdot \Diamond\sim A)$

If A, then it's necessary that B = $(A \supset \Box B)$ or $\Box(A \supset B)$

If A, then it's impossible that B = $(A \supset \Box\sim B)$ or $\Box(A \supset \sim B)$

A *world prefix* is a string of zero or more instances of "W."

A *derived step* is now a line consisting of a world prefix and then "∴" and then a wff.

∴ A	(So A is true in the actual world.)

W ∴ A	(So A is true in world W.)

An *assumption* is now a line consisting of a world prefix and then "asm:" and then a wff.

asm: A	(Assume A is true in the actual world.)

W asm: A	(Assume A is true in world W.)

Modal Inference Rules

First reverse squiggles	~☐A → ◇~A ~◇A → ☐~A	*
and drop diamonds;	◇A → W ∴ A, use a *new* string of W's	*
lastly, drop boxes.	☐A → W ∴ A, use any world prefix	Don't star

*	1	◇(A · B)		Valid
		[∴ ◇A		
*	2	┌ asm: ~◇A		
	3	│ ∴ □~A {from 2}	🡐	reverse squiggles
*	4	│ W ∴ (A · B) {from 1}	🡐	drop diamonds
	5	│ W ∴ A {from 4}		
	6	│ W ∴ B {from 4}		
	7	└ W ∴ ~A {from 3}	🡐	drop boxes
	8	∴ ◇A {from 2; 5 contradicts 7}		

1. Reverse squiggles.
2. Drop initial diamonds, using a new world each time.
3. Lastly, drop each initial box once for each old world. (Never use a new world when you drop a box.)

Drop boxes	$\Box A \quad \rightarrow \quad W \therefore A$, use any world prefix

Drop a box into all worlds with W's.

Drop a box into the actual world just if:

- you have an unmodalized instance of a letter in your original premises or conclusion, or
- you've done everything else possible (including further assumptions if needed) and still have no other worlds.

*	1	◇~H
	2	☐(H ∨ T)
	[∴	☐T
*	3	asm: ~☐T
*	4	∴ ◇~T {from 3}
	5	W ∴ ~T {from 4}
	6	WW ∴ ~H {from 1}
*	7	W ∴ (H ∨ T) {from 2}
*	8	WW ∴ (H ∨ T) {from 2}
	9	W ∴ H {from 5 and 7}
	10	WW ∴ T {from 6 and 8}

Invalid

W	H, ~T
WW	T, ~H

If you can't get a contradiction, construct a refutation.

1 $\Diamond{\sim}H = 1$

2 $\Box(H \vee T) = 1$

$[\therefore \Box T = 0$

Invalid

W	H, ~T
WW	T, ~H

"$\Diamond$A" is true if and only if *at least one world* has "A" true.

"$\Box$A" is true if and only if *all worlds* have "A" true.

If a wff doesn't start with a box or diamond: evaluate each subformula that starts with a box or diamond, and then substitute "1" or "0" for it:

$$\begin{aligned} &{\sim}\Box H \\ &= {\sim}0 \\ &= 1 \end{aligned}$$

$$\begin{aligned} &(\Diamond H \supset \Box T) \\ &= (1 \supset 0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} &{\sim}\Box(H \vee T) \\ &= {\sim}1 \\ &= 0 \end{aligned}$$

*	1	◇(A • B)	Valid
		[∴ ◇A	
*	2	asm: ~◇A	
	3	∴ □~A	{from 2}
*	4	W ∴ (A • B)	{from 1}
	5	W ∴ A	{from 4}
	6	W ∴ B	{from 4}
	7	W ∴ ~A	{from 3}
	8	∴ ◇A	{from 2; 5 contradicts 7}

*	1	◇~H	
	2	□(H ∨ T)	
		[∴ □T	
*	3	asm: ~□T	
*	4	∴ ◇~T	{from 3}
	5	W ∴ ~T	{from 4}
	6	WW ∴ ~H	{from 1}
*	7	W ∴ (H ∨ T)	{from 2}
*	8	WW ∴ (H ∨ T)	{from 2}
	9	W ∴ H	{from 5 and 7}
	10	WW ∴ T	{from 6 and 8}

W	H, ~T
WW	T, ~H

Invalid

1. Reverse squiggles.
2. Drop initial diamonds, using a new world each time.
3. Lastly, drop each initial box once for each old world. (Never use a new world when you drop a box.)

Do these two now:

$$(\Diamond A \supset \Diamond B)$$
$$\therefore \Box(A \supset B)$$

$$(A \cdot \Diamond B)$$
$$\Box((A \cdot B) \supset C)$$
$$\therefore C$$

1. Reverse squiggles.
2. Drop initial diamonds, using a new world each time.
3. Lastly, drop each initial box once for each old world. (Never use a new world when you drop a box.)

If you get stuck, make another assumption by breaking up a wff of the form "$\sim(A \cdot B)$," "$(A \vee B)$," or "$(A \supset B)$."