

$(\sim B \supset (A \cdot C))$   
 $(A \supset \sim C)$   
 $\therefore B$

This is our  
sample  
argument.

# Formal Proofs

From now on, formal proofs will be our main way to test arguments. We'll begin with easier proofs. Our initial strategy for constructing proofs has three steps.

1  $(\sim B \supset (A \cdot C))$

2  $(A \supset \sim C)$

[  $\therefore B$

3 asm:  $\sim B$

[ Block off conclusion

← Assume the opposite

## Step 1: START

Block off the conclusion and add “asm:” followed by the conclusion’s simpler contradictory.

- 1  $(\sim B \supset (A \cdot C))$
- 2  $(A \supset \sim C)$
- [  $\therefore B$
- 3 asm:  $\sim B$

Here the complex wffs are  
1 and 2, both **IF-THENs**.  
You can infer from these if  
you have the first part true  
or the second false.

## Step 2: S&I

Begin the S&I step by glancing at the complex wffs and noticing their forms. You can simplify **AND**, **NOR**, and **NIF** – and you can infer with **NOT-BOTH**, **OR**, and **IF-THEN** if certain other wffs are available.

<p>* 1    <math>(\sim \mathbf{B} \supset (\mathbf{A} \cdot \mathbf{C}))</math>  2    <math>(\mathbf{A} \supset \sim \mathbf{C})</math>  [ <math>\therefore \mathbf{B}</math>  3    asm: <math>\sim \mathbf{B}</math>  4    <math>\therefore (\mathbf{A} \cdot \mathbf{C})</math> {from 1 and 3}    ←</p>	<p>IF-THEN rule:  <math>(\sim \mathbf{B} \supset (\mathbf{A} \cdot \mathbf{C}))</math>  <math>\sim \mathbf{B}</math>  <hr style="width: 50%; margin: 0 auto;"/> <math>(\mathbf{A} \cdot \mathbf{C})</math></p>
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## Step 2: S&I

Go through the complex wffs that aren't starred or blocked off and use these to derive new wffs using S- and I-rules. Star any wff you simplify using an S-rule, or the longer wff used in an I-rule inference.

\* 1  $(\sim B \supset (A \cdot C))$

2  $(A \supset \sim C)$

[  $\therefore B$

3 asm:  $\sim B$

\* 4  $\therefore (A \cdot C)$  {from 1 and 3}

5  $\therefore A$  {from 4}

6  $\therefore C$  {from 4}

AND rule:

$$\frac{(A \cdot C)}{A, C}$$

←

←

## Step 2: S&I

Go through the complex wffs that aren't starred or blocked off and use these to derive new wffs using S- and I-rules. Star any wff you simplify using an S-rule, or the longer wff used in an I-rule inference.

\* 1  $(\sim B \supset (A \cdot C))$

\* 2  $(A \supset \sim C)$

[  $\therefore B$

3 asm:  $\sim B$

\* 4  $\therefore (A \cdot C)$  {from 1 and 3}

5  $\therefore A$  {from 4}

6  $\therefore C$  {from 4}

7  $\therefore \sim C$  {from 2 and 5} ←

IF-THEN rule:

$(A \supset \sim C)$

$A$

---

$\sim C$

## Step 2: S&I

Go through the complex wffs that aren't starred or blocked off and use these to derive new wffs using S- and I-rules. Star any wff you simplify using an S-rule, or the longer wff used in an I-rule inference.

*	1	$(\sim B \supset (A \cdot C))$	Valid
*	2	$(A \supset \sim C)$	
		[ $\therefore B$	
	3	asm: $\sim B$	
*	4	$\therefore (A \cdot C)$ {from 1 and 3}	<div style="display: flex; align-items: center; justify-content: center;"> <div style="font-size: 3em; margin-right: 10px;">[</div> <div style="text-align: left;"> <p>block off from the assumption on down</p> </div> </div>
	5	$\therefore A$ {from 4}	
	6	$\therefore C$ {from 4}	
	7	$\therefore \sim C$ {from 2 and 5}	
	8	$\therefore B$ {from 3; 6 contradicts 7}	<div style="display: flex; align-items: center; justify-content: center;"> <div style="font-size: 2em; margin-right: 10px;">←</div> <div>derive conclusion</div> </div>

## Step 3: RAA

When some pair of not-blocked-off lines contradicts, apply RAA and derive the original conclusion. Your proof is done!

*S- and I-Rules*

AND	$\frac{(P \cdot Q)}{P, Q}$	NOR	$\frac{\sim(P \vee Q)}{\sim P, \sim Q}$	NIF	$\frac{\sim(P \supset Q)}{P, \sim Q}$
NN	$\frac{\sim\sim P}{P}$	IFF	$\frac{(P \equiv Q)}{(P \supset Q), (Q \supset P)}$	NIFF	$\frac{\sim(P \equiv Q)}{(P \vee Q), \sim(P \cdot Q)}$
NOT-BOTH		OR		IF-THEN	
$\frac{\sim(P \cdot Q)}{P}$	$\frac{\sim(P \cdot Q)}{Q}$	$\frac{(P \vee Q)}{\sim P}$	$\frac{(P \vee Q)}{\sim Q}$	$\frac{(P \supset Q)}{P}$	$\frac{(P \supset Q)}{\sim Q}$
$\sim Q$	$\sim P$	$Q$	$P$	$Q$	$\sim P$

*RAA*: Suppose that some pair of not-blocked-off lines has contradictory wffs. Then block off all the lines from the last not-blocked-off assumption on down and infer a line consisting in “ $\therefore$ ” followed by a contradictory of that assumption.



*	1	$(\sim B \supset (A \cdot C))$	←	premises (no “asm” or “∴”)
*	2	$(A \supset \sim C)$	←	
		[ ∴ B		
	3	asm: $\sim B$	←	assumption (“asm”)
*	4	$\therefore (A \cdot C)$ {from 1 and 3}	←	derived lines (“∴”)
	5	$\therefore A$ {from 4}	←	
	6	$\therefore C$ {from 4}	←	
	7	$\therefore \sim C$ {from 2 and 5}	←	
	8	$\therefore B$ {from 3; 6 contradicts 7}	←	

A *formal proof* is a vertical sequence of zero or more premises followed by one or more assumptions or derived lines, where each derived line follows from previously not-blocked-off lines by one of the S- and I-rules listed above or by RAA, and each assumption is blocked off using RAA.

Two wffs are *contradictories* if they are exactly alike except that one starts with an additional “ $\sim$ .”

A *simple wff* is a letter or its negation; any other wff is *complex*.

Valid

- \* 1  $(\sim B \supset (A \cdot C))$
- \* 2  $(A \supset \sim C)$
- [  $\therefore B$
- 3 [ asm:  $\sim B$
- \* 4 [  $\therefore (A \cdot C)$  {from 1 and 3}
- 5 [  $\therefore A$  {from 4}
- 6 [  $\therefore C$  {from 4}
- 7 [  $\therefore \sim C$  {from 2 and 5}
- 8  $\therefore B$  {from 3; 6 contradicts 7}

## Proof Strategy

- 1 **START:** Assume the opposite of the conclusion.
- 2 **S&I:** Derive whatever you can using S- and I-rules, until you get a contradiction.
- 3 **RAA:** Apply RAA and derive the original conclusion.

- 1     $(A \supset B)$
- $[ \therefore (B \supset A)$
- \* 2    asm:  $\sim(B \supset A)$
- 3     $\therefore B$     {from 2}                    We can derive
- 4     $\therefore \sim A$     {from 2}                    nothing further.

## Proof strategy to include invalid arguments:

- 1    **START:** Assume the opposite of the conclusion.
- 2    **S&I:** Derive whatever you can using S- and I-rules.
- 3    **RAA:** If you get a contradiction, apply RAA and derive the original conclusion.
- 4    **REFUTE:** If you don't get a contradiction, construct a refutation box.

- 1  $(A^0 \supset B^1) = 1$
- $[\therefore (B^1 \supset A^0) = 0$
- \* 2 asm:  $\sim(B \supset A)$
- 3  $\therefore B$  {from 2}
- 4  $\therefore \sim A$  {from 2}

Invalid

B,  $\sim A$

Step 4 – REFUTE: If you can’t get a contradiction, then:

- draw a box containing any simple wffs (letters or their negation) that aren’t blocked off;
- in the original argument, mark each letter “1” or “0” or “?” depending on whether you have the letter or its negation or neither in the box;
- if these truth conditions make the premises all true and conclusion false, then this shows the argument to be invalid.

\* 1  $(\sim B \supset (A \cdot C))$  Valid  
 \* 2  $(A \supset \sim C)$   
 [  $\therefore B$   
 3 asm:  $\sim B$   
 \* 4  $\therefore (A \cdot C)$  {from 1 and 3}  
 5  $\therefore A$  {from 4}  
 6  $\therefore C$  {from 4}  
 7  $\therefore \sim C$  {from 2 and 5}  
 8  $\therefore B$  {from 3; 6 contradicts 7}

1  $(A^0 \supset B^1) = 1$  Invalid  
 [  $\therefore (B^1 \supset A^0) = 0$   
 \* 2 asm:  $\sim(B \supset A)$  B,  $\sim A$   
 3  $\therefore B$  {from 2}  
 4  $\therefore \sim A$  {from 2}

- 1 START: Assume the opposite of the conclusion.
- 2 S&I: Derive whatever you can using S- and I-rules.
- 3 RAA: If you get a contradiction, apply RAA and derive the original conclusion.
- 4 REFUTE: If you don't get a contradiction, construct a refutation box.

1  $(B \vee A)$

2  $(B \supset A)$

[  $\therefore \sim(A \supset \sim A)$

3 asm:  $(A \supset \sim A)$



We're stuck!

*Here we get stuck using our old strategy –  
so we need to make another assumption.*

- 1 START: Assume the opposite of the conclusion.
- 2 S&I: Derive whatever you can using S- and I-rules.
- 3 RAA: If you get a contradiction, apply RAA and derive the original conclusion.
- 4 REFUTE: If you don't get a contradiction, construct a refutation box.

1  $(B \vee A)$

2  $(B \supset A)$

[  $\therefore \sim(A \supset \sim A)$

3 asm:  $(A \supset \sim A)$



We're stuck!

We're stuck when:

We can't apply S- or I-rules further.

And we can't prove the argument  
VALID (since we have no contradiction)  
or INVALID (since we don't have enough  
simple wffs for a refutation).

- 1 (B  $\vee$  A)
- 2 (B  $\supset$  A)
- [  $\therefore \sim(A \supset \sim A)$
- 3 asm: (A  $\supset$   $\sim A$ )
- 4 asm: B {break up 1}

When you're stuck,  
try to make another  
assumption.



**ASSUME:** Look for a complex wff that isn't starred or blocked off or broken. This wff will have one of these forms:

NOT-BOTH	$\sim(A \cdot B)$
OR	$(A \vee B)$
IF-THEN	$(A \supset B)$

Assume one side or its negation – and then return to step 2 (S&I).



- 1     $(B \vee A)$
  - \*\* 2     $(B \supset A)$
  - $[\therefore \sim(A \supset \sim A)$
  - \*\* 3    asm:  $(A \supset \sim A)$
  - 4       asm: B    {break up 1 }
  - 5        $\therefore A$     {from 2 and 4}**            ←
  - 6        $\therefore \sim A$     {from 3 and 5}**           ←
- Contradiction!

S&I: Go through the complex wffs that aren't starred or blocked off and use these to derive new wffs using S- and I-rules. Star (*with one star for each live assumption*) any wff you simplify using an S-rule, or the longer wff used in an I-rule inference.

- 1  $(B \vee A)$
- 2  $(B \supset A)$
- [  $\therefore \sim(A \supset \sim A)$
- 3 asm:  $(A \supset \sim A)$
- 4   asm: B   {break up 1}
- 5   [  $\therefore A$    {from 2 and 4}
- 6   [  $\therefore \sim A$    {from 3 and 5}
- 7  $\therefore \sim B$    {from 4; 5 contradicts 6}   ←   Apply RAA.

RAA: If you have a contradiction, apply RAA on the last live assumption. If all assumptions are now blocked off, you've proved the argument valid. *Otherwise, erase star strings having more stars than the number of live assumptions* – and then return to step 2 (S&I).

# Valid

- \* 1  $(B \vee A)$
- 2  $(B \supset A)$
- [  $\therefore \sim(A \supset \sim A)$
- \* 3 asm:  $(A \supset \sim A)$
- 4 [ asm: B {break up 1}
- 5 [  $\therefore A$  {from 2 and 4}
- 6 [  $\therefore \sim A$  {from 3 and 5}
- 7  $\therefore \sim B$  {from 4; 5 contradicts 6}
- 8  $\therefore A$  {from 1 and 7} ←**
- 9  $\therefore \sim A$  {from 3 and 8} ←**
- 10  $\therefore \sim(A \supset \sim A)$  {from 3; 8 contradicts 9} ←

We use “ $\sim B$ ”  
to get a  
contradiction &  
finish the proof.

- \* 1  $(B \vee A)$  Valid
- 2  $(B \supset A)$
- [  $\therefore \sim(A \supset \sim A)$
- \* 3  $\text{asm: } (A \supset \sim A)$
- 4  $\left[ \begin{array}{l} \text{asm: } B \quad \{\text{break up 1}\} \\ \therefore A \quad \{\text{from 2 and 4}\} \\ \therefore \sim A \quad \{\text{from 3 and 5}\} \end{array} \right.$
- 7  $\therefore \sim B \quad \{\text{from 4; 5 contradicts 6}\}$
- 8  $\therefore A \quad \{\text{from 1 and 7}\}$
- 9  $\therefore \sim A \quad \{\text{from 3 and 8}\}$
- 10  $\therefore \sim(A \supset \sim A) \quad \{\text{from 3; 8 contradicts 9}\}$

Strategy:

Start
S&I
RAA
Assume
Refute

- 1  $\sim(A \cdot B)$
- [  $\therefore (\sim A \cdot \sim B)$
- 2 asm:  $\sim(\sim A \cdot \sim B)$       ← Assume opposite.

Then we're stuck!

We can't apply S- or I-rules or RAA; and we don't have enough simple wffs for a refutation.

START: Assume the opposite of the conclusion.

1  $\sim(A \cdot B)$

[  $\therefore (\sim A \cdot \sim B)$

2 asm:  $\sim(\sim A \cdot \sim B)$

3 asm: A {break up 1} ←

When you're stuck,  
try to make another  
assumption.

**ASSUME:** Look for a complex wff that isn't starred or blocked off or broken. This wff will have one of these forms:

NOT-BOTH  $\sim(A \cdot B)$

OR  $(A \vee B)$

IF-THEN  $(A \supset B)$

Assume one side or its negation – and then return to step 2 (S&I).

- \*\* 1     $\sim(A \cdot B)$   
       [  $\therefore (\sim A \cdot \sim B)$   
 2    asm:  $\sim(\sim A \cdot \sim B)$   
 3    asm: A    {break up 1}  
 4     $\therefore \sim B$     {from 1 and 3}     $\leftarrow$     Derive further lines.

We're stuck again! But now all complex wffs are either starred or blocked off or broken.

S&I: Go through the complex wffs that aren't starred or blocked off and use these to derive new wffs using S- and I-rules. Star (*with one star for each live assumption*) any wff you simplify using an S-rule, or the longer wff used in an I-rule inference.

- \*\* 1  $\sim(A^1 \cdot B^0) = 1$   
 [  $\therefore (\sim A^1 \cdot \sim B^0) = 0$   
 2 asm:  $\sim(\sim A \cdot \sim B)$   
 3 asm: A {break up 1}  
 4  $\therefore \sim B$  {from 1 and 3}

Invalid

A,  $\sim B$

REFUTE: Construct a refutation box if you can't apply S- and I-rules or RAA further, and yet all complex wffs are either starred or blocked off or broken.



\* 1 (B  $\vee$  A) Valid  
 2 (B  $\supset$  A)  
 [  $\therefore \sim(A \supset \sim A)$   
 \* 3 asm: (A  $\supset$   $\sim A$ )  
 4 [ asm: B {break up 1}  
 5 [  $\therefore A$  {from 2 and 4}  
 6 [  $\therefore \sim A$  {from 3 and 5}  
 7  $\therefore \sim B$  {from 4; 5 contradicts 6}  
 8  $\therefore A$  {from 1 and 7}  
 9  $\therefore \sim A$  {from 3 and 8}  
 10  $\therefore \sim(A \supset \sim A)$  {from 3; 8 contradicts 9}

\*\* 1  $\sim(A^1 \cdot B^0) = 1$  Invalid  
 [  $\therefore (\sim A^1 \cdot \sim B^0) = 0$   
 2 asm:  $\sim(\sim A \cdot \sim B)$   
 3 asm: A {break up 1}  
 4  $\therefore \sim B$  {from 1 and 3}

A, $\sim B$
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Start	S&I	RAA	Assume	Refute
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*Traditional Copi proofs* use eight inference rules and ten replacement rules. Here are the inference rules:

AD Addition	$\frac{P}{(P \vee Q)}$
CJ Conjunction	$\frac{P \quad Q}{(P \cdot Q)}$
DI Dilemma	$\frac{((P \supset Q) \cdot (R \supset S)) \quad (P \vee R)}{(Q \vee S)}$
DS Disjunctive Syllogism	$\frac{(P \vee Q) \quad \sim P}{Q}$

HS Hypothetical Syllogism	$\frac{(P \supset Q) \quad (Q \supset R)}{(P \supset R)}$
MP Modus Ponens	$\frac{(P \supset Q) \quad P}{Q}$
MT Modus Tollens	$\frac{(P \supset Q) \quad \sim Q}{\sim P}$
SP Simplification	$\frac{(P \cdot Q)}{P}$

## Here are the ten Copi replacement rules:

AS	Association	$(P \vee (Q \vee R)) = ((P \vee Q) \vee R)$ $(P \cdot (Q \cdot R)) = ((P \cdot Q) \cdot R)$
CM	Commutation	$(P \vee Q) = (Q \vee P)$ $(P \cdot Q) = (Q \cdot P)$
DB	Distribution	$(P \cdot (Q \vee R)) = ((P \cdot Q) \vee (P \cdot R))$ $(P \vee (Q \cdot R)) = ((P \vee Q) \cdot (P \vee R))$
DM	De Morgan	$\sim(P \cdot Q) = (\sim P \vee \sim Q)$ $\sim(P \vee Q) = (\sim P \cdot \sim Q)$
DN	Double Negation	$P = \sim\sim P$
EQ	Equivalence	$(P \equiv Q) = ((P \supset Q) \cdot (Q \supset P))$ $(P \equiv Q) = ((P \cdot Q) \vee (\sim P \cdot \sim Q))$
EX	Exportation	$((P \cdot Q) \supset R) = (P \supset (Q \supset R))$
IM	Implication	$(P \supset Q) = (\sim P \vee Q)$
RP	Repetition	$P = (P \vee P)$ $P = (P \cdot P)$
TR	Transposition	$(P \supset Q) = (\sim Q \supset \sim P)$

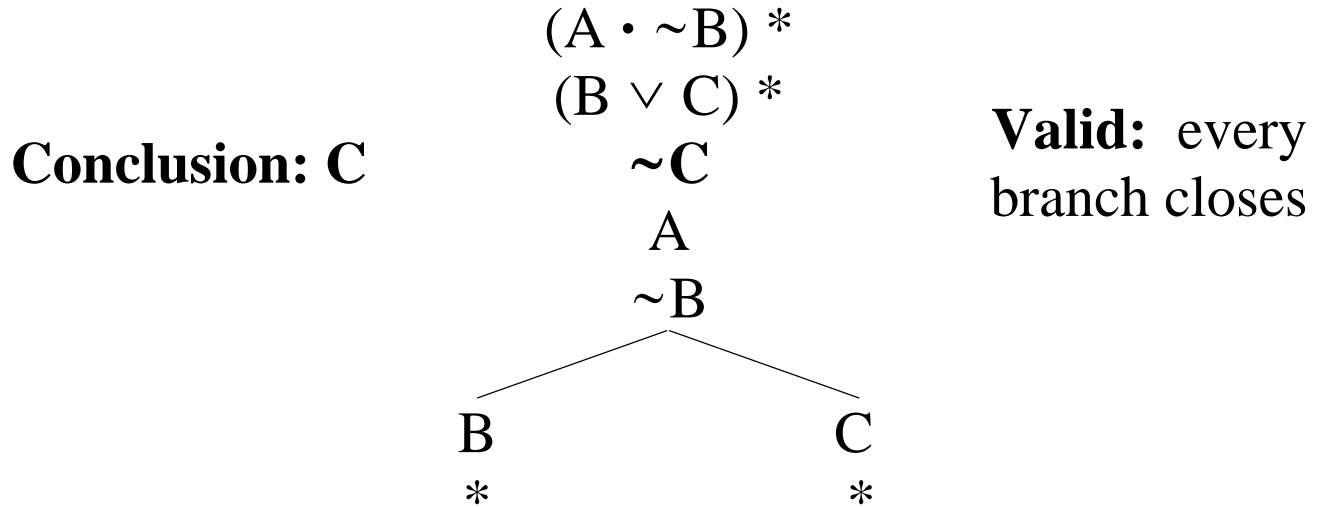
## Conclusion: B

- 1 T
- 2  $(T \supset (B \vee M))$
- 3  $(M \supset H)$
- 4  $\sim H$
- 5  $(B \vee M)$  {MP 1+2}
- 6  $\sim M$  {MT 3+4}
- 7  $(M \vee B)$  {CM 5}
- 8 B {DS 6+7}

Many Copi proofs directly derive the conclusion from the premises. Copi also provides for conditional and indirect (RAA) proofs.

CP Conditional Proof	If you assume P and later derive Q, then you can star all the lines from P to Q [showing that you aren't to use them to derive further steps] and then derive $(P \supset Q)$ .
RA Reductio ad Absurdum	If you assume P and later derive $(Q \cdot \sim Q)$ , then you can star all the lines from P to $(Q \cdot \sim Q)$ [showing that you aren't to use them to derive further steps] and then derive $\sim P$ .

*Truth trees* break formulas into the cases that make them true. Here's a truth tree for “ $(A \cdot \sim B), (B \vee C) \therefore C$ ”:



An argument is valid if and only if every branch eventually *closes* (has a self-contradiction).